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COMMENT

Resistor–diode percolation on the hierarchical diamond lattice

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Abstract. The resistor-diode percolation problem on the hierarchical diamond lattice is studied by using an exact real space renormalisation group transformation. This problem exhibits several critical and multicritical points. The flow diagram obtained is topologically equivalent to the one found by Redner in the framework of an approximate renormalisation group transformation on regular lattices.

General percolation models are interesting because they exhibit a rich structure and model several problems in physics (Broadbent and Hammersley 1957, Stauffer 1979, Essam 1980). Among those the diode-resistor model exhibits novel features driven by diode-resistor concentration and diode orientation. This problem has been studied for regular lattices using an approximate position space renormalisation group (Redner 1982a, Dorogovtsev 1982). It is interesting to see if the study of this problem on a lattice for which an exact renormalisation group can be devised leads to the same qualitative features found in the above works.

Recently, hierarchical lattices have been widely studied (Berker and Ostlund 1979, Kaufman and Griffiths 1984). Renormalisation group transformations which are approximate for regular lattices are exact for such hierarchical lattices. It is then natural to investigate the diode-resistor model on the hierarchical diamond lattice.

The hierarchical diamond lattice is constructed by repeatedly replacing each single bond by a cell with four bonds (see figure 1(a)). There are three differents types of bonds: a resistor-like bond that connects two neighbouring sites in both directions (ij and ji directions in figure 1(b)) and two diode-like bonds that makes the connection only in one direction (ij or ji). Each bond may be vacant with independent probability

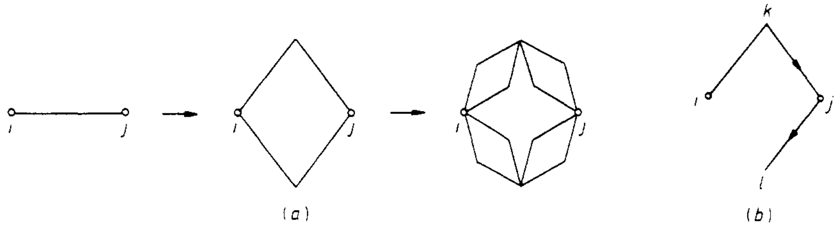

Figure 1. (a) Construction of the diamond lattice; each single bond is repeatedly replaced by four new bonds. (b) A cell with a vacant bond (il), a resistor-like bond (ik), a ij diode-like bond (kj) and a ji diode-like bond (jl).

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q or occupied as follows: by a resistor with probability z, an ij oriented diode with probability x and finally an ji oriented diode with probability y. Of course, we have $q+z+x+y=1$. We want to study the probability of connecting the boundary sites i and j (see figure 1(a)) (Kaufman and Andelman 1984).

The renormalisation group transformation is defined through a mapping between a cell and a bond (Reynolds *et al* 1977) according to the following rules: (i) every configuration that traverses the cell in both directions (ij and ji) maps to a resistor, (ii) every configuration that traverses the cell in one direction (ij or ji) maps to a diode (ij or ji) and (iii) every configuration that does not traverse the cell maps to a vacant bond. Thus we have the following recursion equations:

$$z' = -z^4 + 2z^2(1+4xy) + 4z(xy^2 + x^2y) + 2x^2y^2 \tag{1}$$

$$x' = -x^4 - 4x^3z + 2x^2(1 - y^2 - 2yz - 3z^2) + 4x(z - zy^2 - 2z^2y - z^3). \tag{2}$$

The recursion equation for y' is obtained from (2) by interchanging x and y. The equation for q is not independent since $q = 1 - x - y - z$.

The flow diagram is drawn in figure 2. We have four trivial fixed points: $(x^*, y^*, z^*) = (0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. They correspond respectively to a lattice completely disconnected or a lattice completely occupied by only one type of bond. Each fixed point controls one of the four following phases: non-percolating, ij or ji diode and isotropic percolating phase. These phases are separated by two planes associated with second-order phase transitions. The intersection of these planes defines

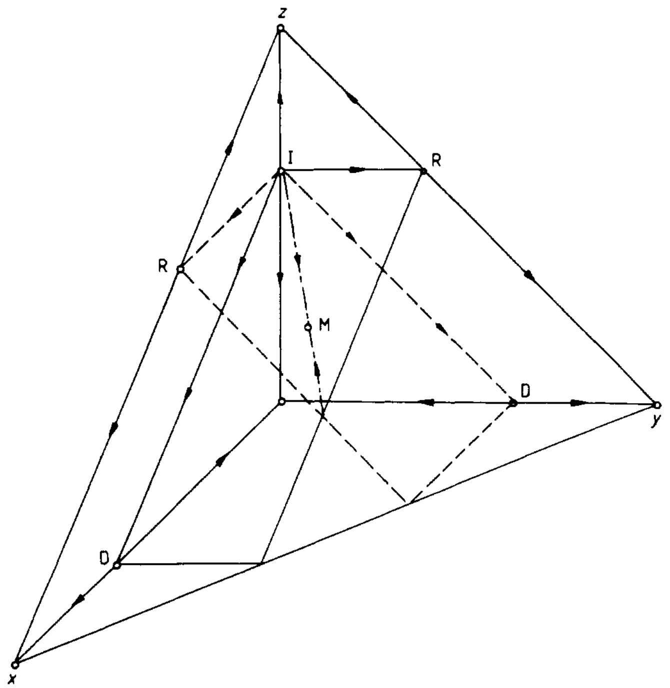

Figure 2. Flow diagram. z is the resistor concentration, x the ij diode concentration and y the ji diode concentration. The non-trivial fixed points are the following: D, directed percolation; R, reverse percolation; I, isotropic percolation and M, mixed percolation. The mixed line is the multicritical line.

a multicritical line. The equations for these planes are

$$x+z-(\sqrt{5}-1)/2=0 \qquad y+z-(\sqrt{5}-1)/2=0. \tag{3}$$

On these planes there are six non-trivial fixed points.

(*a*) The fixed points describing directed percolation (Kinzel and Yeomans 1981). The fixed point $((\sqrt{5}-1)/2, 0, 0)$ describes the outset of an infinite cluster percolating along one direction (ij). The linearised transformation matrix $T_{\alpha\beta}=\partial\alpha/\partial\beta$ ($\alpha=z'$, x'. or y' and $\beta=z$, x or y) has one relevant eigenvalue $\lambda_1=2(3-\sqrt{5})$ conjugated to the eigenvector (1, 0, 0) and a doubly degenerate irrelevant eigenvalue $\lambda_2=0$. The x direction is unstable and all others are stable. This fixed point attracts all the points on the surface which sepatates the non-percolating phase from the (ij) diode phase. The properties of the fixed point $(0, (\sqrt{5}-1)/2, 0)$ are similar.

(*b*) The fixed point describing isotropic percolation. For the bond percolation the fixed point $(0, 0, (\sqrt{5}-1)/2)$ is a simple critical point (Kaufman and Andelman 1984). Here, it is a multicritical point describing a transition in which an infinite cluster forms isotropically. The linearised transformation has a triply degenerate relevant eigenvalue $\lambda=2(3-\sqrt{5})$. This point is unstable in all directions. It is a fourth-order critical point.

(*c*) The fixed point describing reverse percolation. The fixed point $(1-(\sqrt{5}-1)/2$, $0, (\sqrt{5}-1)/2)$ describes a transition from a diode phase, in which there is connectivity along one direction (ij), to the resistor phase in which isotropic connectivity occurs. The linearised transformation has one relevant eigenvalue $\lambda_1=2(3-\sqrt{5})$ conjugated to the eigenvector $(-1, 0, 1)$ and a doubly degenerate irrelevant eigenvalue $\lambda_2=0$. This fixed point is unstable in the direction $x=-z$ and stable in the other directions. This fixed point is an attractor for all points on the surface separating the resistor and the (ij) diode phase. The properties of the fixed point $(0, 1-(\sqrt{5}-1)/2, (\sqrt{5}-1)/2)$ are similar.

(*d*) The fixed point describing the mixed percolation $(\sqrt{5}-2, \sqrt{5}-2, 1-(\sqrt{5}-1)/2)$. Here there is a transition in which an infinite isotropic cluster forms with resistors and diodes. The linearised transformation has a doubly degenerate relevant eigenvalue $\lambda_1=2(3-\sqrt{5})$ and an irrelevant one $\lambda_2=14-6\sqrt{5}$, conjugated to the eigenvector (1, 1, -1). This point is an attractor for all points on the multicritical line, where the four phases meet. This line starts at the point $(1-(\sqrt{5}-1)/2)$, $1-(\sqrt{5}-1)/2$, $\sqrt{5}-2)$ on the plane $z+x+y=1$ and ends up at the fixed point of isotropic percolation $(0, 0, (\sqrt{5}-1)/2)$.

Note that on the diamond lattice the isotropic and directed fixed points have similar critical properties (critical concentrations and exponents). The multicritical line expresses the fact that the percolating cluster can be formed by using all the possible types of bonds.

In conclusion we see that this simple position space renormalisation group transformation allows us to describe exactly a complex physical situation with critical and multicritical behaviour.

Finally, the flow diagram obtained for the diamond lattice is topologically equivalent to the one obtained by Redner (1982a) in the framework of an approximate renormalisation group transformation on regular lattices. For the square lattice, Redner (1982b) showed that the correlation length exponents are constant along the multicritical line. Moreover, for any lattice, he showed that, at the isotropic percolation threshold, the correlation length diverges with the same exponent independent from the approach to the transition within the plane $x=y$. The same properties hold in this hierarchical model.

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